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L_p Rational Approximation

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We shall consider here the following approximation problem. Let X denote a compact Hausdorff space, μ a regular Borel measure on X, and let $\{g_1, \ldots, g_n\}$ and $\{h_1, \ldots, h_m\}$ be fixed sets of linearly independent real-valued continuous functions on X. Furthermore, let $\mathscr{N} = \text{span}\{g_1, \ldots, g_n\}$, $\mathscr{D} = \text{span}\{h_1, \ldots, h_m\}$, and $R^+ = \{N/D \mid N \in \mathscr{N}, D \in \mathscr{D}$ and $D(x) > 0$ for all $x \in X\}$. Assume R^+ is nonvoid. Then given $f \in L_p(X, \mu)$ with $1 \leqslant p < \infty$, does there exist an $r_0 \in R^+$ such that $\| f - r_0 \|_p = \inf_{r \in R^+} \| f - r \|_p$?

As the following example shows, the answer to this question is negative in general.

EXAMPLE 1. Let $X = [0, 1]$, $p \geqslant 1$, μ the Lebesgue measure on $[0, 1]$, $f(x) = x^{-\frac{1}{2}p}$ and $R^+ = \{a/(b + cx^{\frac{1}{2}p}) \mid b + cx^{\frac{1}{2}p} > 0$ for all $x \in [0, 1]\}$. Then the sequence $\{r_n\} = \{1/(x^{\frac{1}{2}p} + n^{-1})\} \subset R^+$ satisfies $\| f - r_n \|_p \to 0$. (To see this, note that $| r_n - f | = 1/x^{\frac{1}{2}p}(1 + nx^{\frac{1}{2}p})$ is monotone decreasing and pointwise convergent to zero on $(0, 1]$.) But $f \notin R^+$ so that f has no best approximation in R^+. Clearly, this R^+ has the defect that it is not a closed subset of $L_p[0, 1]$.

Under an assumption given below we will explicitly determine the closure of R^+ in the appropriate L_p space, and show that best approximations always exist in the closure. As a corollary we will be able to show that a sufficient condition given by Cheney and Goldstein in [1] (and generalized by Dunham in [3]) for the existence of best approximations in R^+ is, in fact, both necessary and sufficient (after a slight weakening). The results presented here are from the author's doctoral thesis [4].

The analysis will be carried out under the following assumption on the measure space (X, μ) and the set $\mathscr{D}$. (See [2] and [3] also.)

Assumption $(*)$. If $D \in \mathscr{D}$ is such that $D \not\equiv 0$, then $\mu(Z(D)) = 0$ where $Z(D) = \{x \in X \mid D(x) = 0\}$.

Note that (*) excludes the important case when X is a discrete set and μ is counting measure. We shall consider this problem in a forthcoming paper.

DEFINITION 1. For arbitrary but fixed p with $1 \leqslant p \leqslant \infty$, R shall denote the set $\{N/D \mid N \in \mathcal{N}, D \in \mathcal{D}, D(x) \geqslant 0$ for all $x \in X$ and $\int_X |N/D|^p \, d\mu < \infty\}$.

Note that by (*) every element of the form N/D with $D \not\equiv 0$ is defined except perhaps on a set of measure zero so that R may be considered to be a subset of $L_p(X, \mu)$ (which we henceforth shorten to L_p).

Also it is clear that the set R depends on p. Let this dependence be denoted by R_p. Since $L_p(X, \mu) \subset L_q(X, \mu)$ when $p > q$ and X is a finite measure space, it follows that $R_p \subset R_q$ if $p > q$. Moreover, Example 1 above shows that, in general, the containment is strict since in that case $x^{-(1/2p)} \in R_p \sim R_q$ for any $q \geqslant 2p$. Of course the set $\bigcap_{p=1}^{\infty} R_p$ is nonvoid since it contains R^+ which is nonvoid by hypothesis. On the other hand, if $R_1 \subset L_\infty(X, \mu)$ (as is the case for polynomial rational functions on [0, 1]) then it is clear that $R_p = R_q$ for all $p, q \geqslant 1$.

DEFINITION 2. Let E be a normed linear space. A subset $M \subset E$ is called boundedly weakly sequentially compact (b.w.s.c.) if every bounded sequence in M admits a subsequence converging to some element of M with respect to the weak topology on E (see [6, p. 121]).

Remark 1. Using standard arguments (see [7, p. 97, Corollary 2.2] for example, noting that b.w.s.c. is all that is needed in the proof) it is readily shown that a b.w.s.c. subset M of a normed linear space E always has the property that each element of E has a best approximation in M. As will be seen, b.w.s.c. seems to be the strongest compactness property satisfied by the set R for $1 < p < \infty$.

The proofs of the following lemmas are simple modifications of arguments found in [1] and have thus been omitted. In each case p is arbitrary but fixed with $1 \leqslant p < \infty$.

LEMMA 1. *Assume (*) holds and let $\{r_j\}$ be a sequence in R such that $\{\| r_j \|_p\}$ is bounded and $\| D_j \|_\infty = 1$ for all j where $r_j = N_j/D_j$. Then $\{\| N_j \|_\infty\}$ is also bounded.*

LEMMA 2. *Assume (*) holds and let $\{r_j\} \subset R$ be bounded. Then there exists a subsequence $\{r_{j_k}\}$ and an $r \in R$ such that $r_{j_k} \to r$ uniformly on each closed subset of a set whose complement has measure zero. In particular, $r_{j_k} \to r$ μ-almost everywhere (μ.a.e.).*

We now have the following:

THEOREM. *Assume (*) holds and let p be arbitrary but fixed with $1 \leqslant p < \infty$. Then*

(a) *R is the closure of R^+ with respect to the norm topology on L_p.*

(b) *Each $f \in L_p$ has a best approximation in R.*

(c) *If $p > 1$ then R is b.w.s.c. in L_p and is the weak sequential closure of R^+ (i.e., $R = \{\varphi \in L_p \mid$ there exists a sequence in R^+ converging to φ with respect to the weak topology$\}$).*

Proof. (a) Let $\{r_j\}$ be a sequence in R and suppose $f \in L_p$ is such that $\| f - r_j \|_p \to 0$. Then $\{r_j\}$ is bounded and by Lemma 2 there is a subsequence (which we do not relabel) and an $r \in R$ such that $r_j \to r$ μ.a.e. and hence in measure also. But $r_j \to f$ in measure [5, p. 201] and so $f = r$. Thus R is closed and hence contains the closure of R^+.

To obtain the reverse inclusion, let $r = N/D \in R$ be arbitrary where $D(x) \geqslant 0$ for all $x \in X$. Define $\{r_j\} \subset R^+$ by $r_j = N_j/D_j$ where $N_j = N$ and $D_j = D + h/j$ for $j = 1, 2, \ldots$ where $h \in \mathscr{D}$ is such that $h(x) > 0$ for all $x \in X$ (h exists since $R^+ \neq \varnothing$). Then $r_j \to r$ μ.a.e. and the inequality

$$| r_j(x) - r(x)|^p = | N(x)\, h(x)/(jD^2(x) + D(x)\, h(x))|^p \leqslant | N(x)/D(x)|^p$$

(μ.a.e.) together with the Lebesgue dominated convergence theorem implies that $\| r_j - r \|_p \to 0$. Thus (a) is proved.

(b) Let $f \in L_p$ and choose $\{r_j\} \subset R$ such that $\| f - r_j \|_p \to \inf_{r \in R} \| f - r \|_p$. As in (a), a subsequence of $\{r_j\}$ converges μ.a.e. to some $r \in R$ and applying Fatou's lemma we conclude that r is a best approximation to f from R.

(c) The proof of (c) follows directly from Lemma 2 and the fact that for $1 < p < \infty$, if a sequence in L_p is bounded and converges in measure then it converges with respect to the weak topology to the same limit [5, p. 207]. Q.E.D.

An immediate corollary follows:

COROLLARY. *If $(*)$ holds then each $f \in L_p$ has a best approximation in R^+ if and only if R^+ is closed (i.e., if and only if $R = R^+$).*

Remark 2. A suggestive interpretation of the corollary is that R^+ is closed if and only if whenever $\int_X | N/D |^p \, d\mu < \infty$ with $D(x) \geqslant 0$ for all $x \in X$, then there exists $N_0/D_0 \in R^+$ such that $N_0/D_0 = N/D$ μ.a.e. Thus "singularities" of N/D are "removable."

Remark 3. In [1] Cheney and Goldstein give the following condition on R^+ guaranteeing the existence of best approximations.

Condition. If $N \in \mathscr{N}$ and $D \in \mathscr{D}$ and $f \in L_p$ are such that

$$\left[\int_S | f - N/D |^p \, d\mu\right]^{1/p} \leqslant t$$

for all closed subsets $S \subset Z(D)^c$ then there exists $N_0 \in \mathcal{N}$ and $D_0 \in \mathcal{D}$ with $D_0(x) > 0$ for all $x \in X$ such that $\| f - N_0/D_0 \|_p \leqslant t$.

If we weaken this condition by only requiring it to hold for $D \in \mathcal{D}$ with $D(x) \geqslant 0$ for all $x \in X$ then the condition is both necessary and sufficient for the existence of best approximations in R^+ [we still assume (*) holds]. To see this, first note that since

$$\sup_{\substack{S \subset Z(D)^c \\ S \text{ closed}}} \int_S | f - N/D |^p \, d\mu = \int_X | f - N/D |^p \, d\mu$$

the condition is equivalent to "If $r \in R$ is such that $\| f - r \|_p \leqslant t$ then there is an $r_0 \in R^+$ such that $\| f - r_0 \|_p \leqslant t$." Now suppose R^+ is closed (i.e., best approximations always exist in R^+). Then by Remark 1, for any $r \in R$ there is an $r_0 \in R^+$ such that $r = r_0$ μ.a.e. so that the condition holds. To prove sufficiency (as in [1]) let $t = \inf_{r \in R} \| f - r \|_p = \inf_{r \in R^+} \| f - r \|_p$.

Remark 4. Dunham [3] has independently and simultaneously arrived at existence results similar to (b) of the theorem for a generalized norm defined by $\sigma(f) = \int_X \rho(f) \, d\mu$ where ρ is some nonnegative continuous function on the real line and where f is a bounded measurable function on X. He also obtains (with some restrictions on ρ) a sufficient condition completely analogous to the one of Remark 3. The extension of the results of this paper to that more general setting will be considered in a later paper.

Remark 5. Blatter [2] showed that if one considers the family $R^* \cap L_p$ where $R^* = \{N/D \mid N \in \mathcal{N}, D \in \mathcal{D}\}$ then $R^* \cap L_p$ is approximatively compact for $1 < p < \infty$. That is, if $f \in L_p$ and $\{r_j\} \subset R^* \cap L_p$ are such that $\| f - r_j \|_p \to \inf_{r \in R^* \cap L_p} \| f - r \|_p$ then some subsequence of $\{r_j\}$ converges in the norm topology to some element of $R^* \cap L_p$. A trivial modification of this result yields (b) of the theorem for the case $1 < p < \infty$.

Also, for $1 < p < \infty$ it is clear that (b) is a simple consequence of (c) in the theorem. However, as the following example shows, R is not necessarily b.w.s.c. if $p = 1$. The example also shows that for $1 \leqslant p \leqslant \infty$ R is not boundedly compact so that no strengthening of part (c) of the theorem seems possible.

EXAMPLE 2. Let $X = [0, 1]$, μ be Lebesque measure, and $R = R^+ = R_m{}^n[0, 1] \equiv \{(a_0 + \cdots + a_j x^j)/(b_0 + \cdots + b_k x^k) \mid j \leqslant n,\ k \leqslant m$ and $b_0 + \cdots + b_k x^k > 0$ for all $x \in [0, 1]\}$ where $m \geqslant 1$. Let $1 < p < \infty$ be given and define $\{r_j\} \subset R^+$ by $r_j = j/(j^p x + 1)$. Then $\| r_j \|^p = \int_0^1 j^p/(j^p x + 1) \, dx = (p - 1)^{-1}(1 - (j^p + 1)^{1-p}) \to (p - 1)^{-1} > 0$ while $r_j(x) \to 0$ except for $x = 0$. But since $\| r_j \|_p \to 0$, no subsequence could converge (in norm) to zero which is the only possible limit. Thus R^+ is not boundedly compact if $1 < p < \infty$.

Similarly the sequences $1/(jx + 1)$ and $j/(\log(1 + j)(jx + 1))$ are counter examples for $p = \infty$ and $p = 1$, respectively. The latter example also shows that R is not b.w.s.c. in L_1, in general, since no subsequence of $j/(\log(1 + j)(jx + 1))$ converges to zero with respect to the weak topology; otherwise $1 = \int_0^1 j/(\log(1 + j) \cdot (jx + 1))\, dx \to 0$—a contradiction.

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